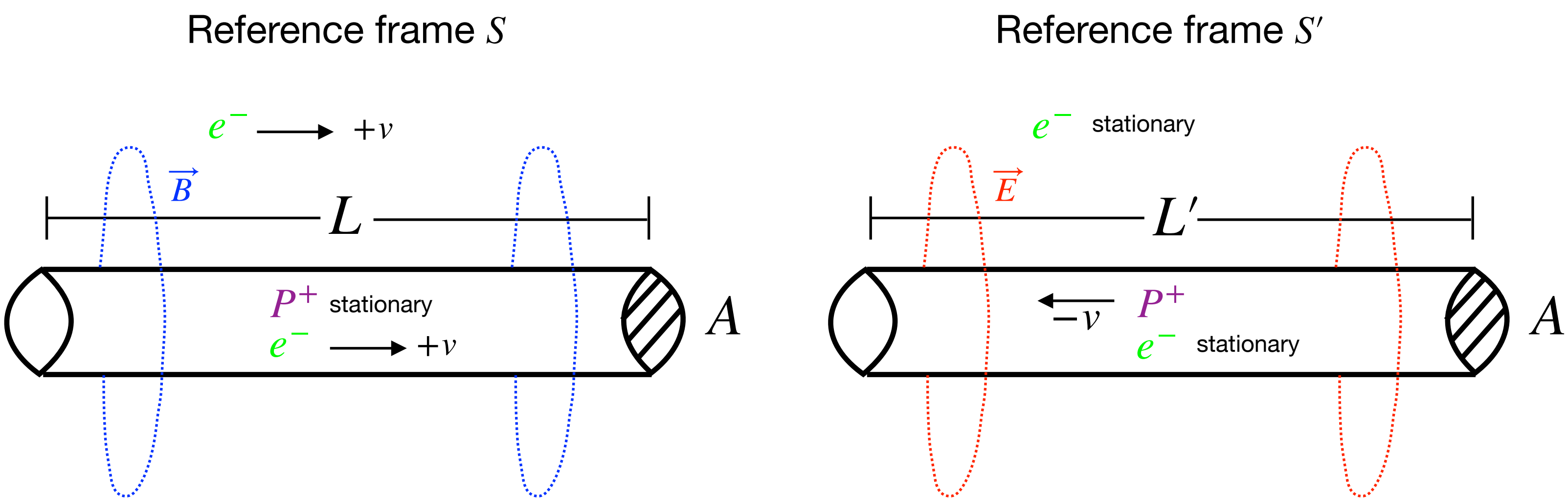

In reference frame S, we have an observer (call her Observer S) who sees a wire with fixed protons and mobile electrons, moving to the right at velocity , v, creating a current, which by convention, moves to the left (in the direction of positive charge). We fire in a test electron which our observer describes as moving to the right at velocity v, the same velocity as the electrons in the wire.

In reference frame S', our observer (call him Observer S') takes the viewpoint of the test electron which Observer S would say is traveling to the right at the same velocity as the mobile electrons in the wire. Observer S' considers himself to be at rest and the electrons in the wire to be at rest. However, he sees the protons in the wire, which Observer S considers stationary, as moving with velocity v to the left.

For this discussion, we'll consider finite lengths of wire. We previously showed that Observer S sees a magnetic field caused by moving electrons in the wire. This magnetic field causes a force that pulls the test electron toward the wire. Because the test electron already has a rightward horizontal velocity. These factors lead to the test electron following a parabolic course to the wire. That discussion can be found here. From that discussion, we found that:

$$|\vec{F}| = \left(\frac{1}{2\pi\epsilon_0 c^2}\left(\frac{I}{r}\right)\right) q v \qquad \text{eq. (1)}$$

where

$\overrightarrow{F}$ is the force on the charged particle
ϵ_0 is the permittivity of free space
c is the speed of light
I is the current in the wire
r is the distance from the wire to the test particle
q is the electric charge of the test particle
v is the speed of the test particle

The current, I, in the length of wire in S can be viewed as the amount of charge per unit volume that passes through the length of wire, L, through cross sectional area, A, during time Δt:

$$I = \rho_{(-)} v A \qquad \text{eq (2)}$$

where

$\rho_{(-)}$ is the charge density of the electrons in the wire
v is the velocity of the electrons
A is the cross-sectional area of the wire

This makes sense when we analyze units:

$$I = \frac{\text{Charge}}{\text{Length}^3} \cdot \frac{\text{Length}}{\text{Time}} \cdot \text{Length}^2 = \frac{\text{Charge}}{\text{Time}}$$

We replace I in eq (1) with eq (2):

$$|\vec{F}| = \left(\frac{1}{2\pi\epsilon_0 c^2} \left(\frac{\rho_{(-)} v A}{r} \right) \right) q v = \frac{q}{2\pi\epsilon_0} \cdot \frac{\rho_{(-)} A}{r} \cdot \frac{v^2}{c^2} \qquad \text{eq (3)}$$

We know that the total charge, Q, in the wire of of length L in frame S must be (charge density) x (wire length) x (area):

$$Q = \rho L A$$

Likewise, the total charge in frame S' is

$$Q = \rho' L' A$$

We know that the total charge in both frames must be equal. Thus,

$$\rho L A = \rho' L' A \qquad \text{so}$$

$$\rho' = \frac{\rho L}{L'} \qquad \text{eq (4)}$$

Per special relativity, L and L' are related in the following way:

$$L' = L\sqrt{1 - \frac{v^2}{c^2}} \qquad \text{eq (5)}$$

Therefore,

$$\rho' = \frac{\rho}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{eq (6)}$$

To Observer S', the protons in the wire are moving and are subject to length contraction. Thus,

$$\rho'_{(+)} = \frac{\rho_{(+)}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{eq (7)}$$

To Observer *S*, the electrons in the wire are moving and appear length contracted. Therefore,

$$\rho_{(-)} = \frac{\rho'_{(-)}}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \rho'_{(-)} = \rho_{(-)}\sqrt{1 - \frac{v^2}{c^2}} \qquad \text{eq (8)}$$

The total charge density in the wire is going to be the sum of the positive and negative charge densities:

$$\begin{aligned} \rho' &= \rho'_{(+)} + \rho'_{(-)} \\ &= \frac{\rho_{(+)}}{\sqrt{1 - \frac{v^2}{c^2}}} + \rho_{(-)}\sqrt{1 - \frac{v^2}{c^2}} \qquad \text{eq (9)} \end{aligned}$$

In frame *S*, the wire is electrically neutral which means that

$$\rho'_{(+)} = \rho'_{(-)} \qquad \text{eq (10)}$$

Thus,

$$\rho' = \frac{\rho_{(+)}}{\sqrt{1-\frac{v^2}{c^2}}} - \rho_{(+)}\sqrt{1-\frac{v^2}{c^2}}$$

$$= \frac{\rho_{(+)}}{\sqrt{1-\frac{v^2}{c^2}}} - \rho_{(+)}\sqrt{1-\frac{v^2}{c^2}} \cdot \frac{\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{\rho_{(+)} - \rho_{(+)}(1-\frac{v^2}{c^2})}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{eq (11)}$$

$$= \frac{\rho_{(+)}}{\sqrt{1-\frac{v^2}{c^2}}}\left[1-(1-\frac{v^2}{c^2})\right]$$

$$= \frac{v^2}{c^2} \cdot \frac{\rho_{(+)}}{\sqrt{1-\frac{v^2}{c^2}}}$$

In other words, in frame S', the total charge density is not zero as in the neutral wire. Instead, in frame S', the wire is positively charged 1) because of the length contraction of the wire's protons, making them bunch closer together, increasing their density, and 2) because of loss of length contraction in the wire's electrons, making them spread out, decreasing their density.) This positive charge creates an electrical field around the wire which decreases inversely by the distance from the wire. This is expressed mathematically in Gauss's Law:

$$E' = \frac{\rho' A}{2\pi\epsilon_0 r} \qquad \text{eq (12)}$$

We derived an expression for ρ'. When we plug eq (11) it into eq (12), we get:

$$E' = \frac{v^2}{c^2} \cdot \frac{\rho_{(+)}}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{A}{2\pi\epsilon_0 r} \qquad \text{eq (13)}$$

Now, what is the force on our test electron created by this electric field? Recall the Lorentz force law:

$$\vec{F} = q\left(\vec{E} + (\vec{v} \times \vec{B})\right)$$

In frame S', the velocity of the test electron is zero so there is no magnetic force on the test electron. We are left with:

$$\vec{F} = q\vec{E}.$$

We can substitute eq (13) into this expression. Doing so gives us the electrical force on our test electron in frame S':

$$F' = q \cdot \frac{v^2}{c^2} \cdot \frac{\rho}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{A}{2\pi\epsilon_0 r} \qquad \text{eq (14a)}$$

In eq (14a), we've dropped the subscript on ρ because, although the charge matters in determining direction of force when we're dealing with vectors, here we're dealing with just the magnitude of that vector.

OK, so now we've got an expression for the electrical force on our test electron in frame S'. How does that compare withe the magnetic force on the test electron seen in frame S? We derived an expression for that in eq (3). For convenience, I'll reproduce it here:

$$|\vec{F}| = = \frac{q}{2\pi\epsilon_0} \cdot \frac{\rho A}{r} \cdot \frac{v^2}{c^2} \qquad \text{eq (3)}$$ (again, we've dropped the subscript on $\rho_{(-)}$ for the same reason we recently described)

Now let's compare eq (3) with eq (14a). I'll rearrange eq (14a) to make the comparison easier:

$$|\overrightarrow{F'}| == \frac{q}{2\pi\epsilon_0} \cdot \frac{\rho A}{r} \cdot \frac{v^2}{c^2} \cdot \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{eq (14b)}$$

What we find is that:

$$|\overrightarrow{F'}| = \frac{|\overrightarrow{F}|}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{eq (15)}$$

But this isn't exactly what we had hoped for. Remember, Einstein's first postulate of special relativity states that physics should be the same in all inertial frames of reference. This would mean that $|\overrightarrow{F'}| = |\overrightarrow{F}|$. However, this doesn't seem to be the case. How can we reconcile this?

The key is to recognize that one definition of the force is change in momentum per change in time:

$$\overrightarrow{F} = \frac{dp}{dt} \qquad \text{where}$$

dp is an infinitesimal change in momentum
dt is an infinitesimal change in time

From this, we can write:

$$\Delta P_y = F\Delta t \quad \text{and} \quad \Delta P'_y = F'\Delta t' \qquad \text{eq (16)}$$

where

ΔP_y is the change in magnitude of downward momentum of the test electron in frame S
F is the magnitude of the downward force on the test electron in frame S
Δt is the time it takes for the test electron to move from its initial position to the wire as a result of the force acting on it in frame S

And$\Delta P_y'$, F' and $\Delta t'$ are the same quantities in frame S'

Because of time dilatation in special relativity, we know that:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{eq (17)}$$

Manipulating eq (16) and eq (17) we find that:

$$\frac{\Delta P_y'}{\Delta P_y} = \frac{F'\Delta t'}{F\Delta t} = \frac{\left(\frac{F}{\sqrt{1 - \frac{v^2}{c^2}}}\right)\left(\Delta t\sqrt{1 - \frac{v^2}{c^2}}\right)}{F\Delta t} = 1 \quad \Rightarrow \quad \Delta P_y = \Delta P_y' \qquad \text{eq (18)}$$

This means that the motion of the electron in frame S and frame S' is the same, which is what we would expect if Einstein's first postulate of special relativity is true.