This derivation is adapted from "Is the Affine Connection a Tensor" uploaded by Andrew Dotson, 4 June 2019, https://www.youtube.com/watch?v=amoV0Hjnb-Q

When we attempt to perform a coordinate transformation on the derivative of tensors, in general, we get the term that we'd expect if we just took the derivative of components, but in addition, we have a second-derivative "correction" term. Here are examples:

If we take the time derivative of a position 4-vector (to create a velocity 4-vector) we have:

$$\frac{dX'^i}{dt} = \frac{\partial X'^i}{dx^j}\frac{dx^j}{dt} + X^i\frac{\partial^2 X'^i}{\partial x^j \partial^k}\frac{dx^k}{dt} \qquad \text{eq (1)}$$

And if we consider the coordinate transformation, of an acceleration vector, from a free-form falling frame of reference to that of an observer on the ground watching the free-falling object, we find:

In the free-falling frame of reference:

$$\frac{d^2X^\alpha}{d\tau^2} = 0 \qquad \text{eq (2)}$$

From the ground:

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0 \ \text{ where } \ \Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial X^\alpha}\frac{\partial^2 X^\alpha}{\partial x^\mu \partial x^\nu} \qquad \text{eq (3)}$$

Second-derivative terms like $X^i\frac{\partial^2 X'^i}{\partial x^j \partial^k}\frac{dx^k}{dt}$ and $\frac{\partial x^\lambda}{\partial X^\alpha}\frac{\partial^2 X^\alpha}{\partial x^\mu \partial x^\nu}$, are zero if we take transformations in flat space, where basis vectors are the same everywhere. But if we make transformations in curved space (like we encounter, for example, in general relativity) then such second-derivative terms are not zero.

Our goal here is to see if the affine connection is a tensor. The way we'll do this is to compare the affine connections (Γ_1 versus Γ_2) used in transformation of position vectors in two different positions:
$X^\alpha \xrightarrow{\Gamma_1} X^\mu$ versus $X^\alpha \xrightarrow{\Gamma_2} X^{\mu'}$

We'll start with an affine connection in a primed coordinate:

$$\Gamma^{\lambda'}_{\mu'\nu'} = \Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x^\lambda}{\partial X^\alpha}\frac{\partial^2 X^\alpha}{\partial x^\mu \partial x^\nu} \qquad \text{eq (4)}$$

We want to transform $\frac{\partial x^{\lambda\prime}}{\partial X^\alpha}\frac{\partial^2 X^\alpha}{\partial x^{\mu'}\partial x^{\nu\prime}}$ into $\frac{\partial x^\lambda}{\partial X^\alpha}\frac{\partial^2 X^\alpha}{\partial x^\mu \partial x^\nu}$

We know that X^α and $x^{\mu'}$ are functions of unprimed variables:

$X^\alpha = X^\alpha(x^\sigma)$ and $x^\mu = x^{\mu'}(x^\rho)$

So we'll have to use the chain and product rules when performing our transformation. Thus:

$$\begin{aligned}
\Gamma'^{\lambda}_{\mu\nu} &= \frac{\partial x^{\lambda\prime}}{\partial X^\alpha}\frac{\partial^2 X^\alpha}{\partial x^{\mu'}\partial x^{\nu\prime}} \\
&= \left(\frac{\partial x^{\lambda\prime}}{\partial x^\rho}\frac{\partial x^\rho}{\partial X^\alpha}\right)\frac{\partial}{x^{\mu'}}\left(\frac{\partial X^\alpha}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial x^{\nu'}}\right) \\
&= \left(\frac{\partial x^{\lambda\prime}}{\partial x^\rho}\frac{\partial x^\rho}{\partial X^\alpha}\right)\left(\frac{\partial x^\sigma}{\partial x^{\nu'}}\frac{\partial^2 X^\alpha}{\partial x^\sigma \partial x^\tau}\frac{\partial x^\tau}{\partial x^{\mu'}} + \frac{\partial X^\alpha}{\partial x^\sigma}\frac{\partial^2 X^\sigma}{\partial x^{\mu'}\partial x^{\nu'}}\right) \qquad \text{eq (5)}
\end{aligned}$$

By applying the distributive law of multiplication to eq (5), we get 2 terms. We'll work on the first term first:

$$\text{Term 1} = \left(\frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial X^{\alpha}}\right)\left(\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\frac{\partial^2 X^{\alpha}}{\partial x^{\sigma}\partial x^{\tau}}\frac{\partial x^{\tau}}{\partial x^{\mu'}}\right)$$

$$= \frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x^{\mu'}}\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\underbrace{\frac{\partial x^{\rho}}{\partial X^{\alpha}}\frac{\partial^2 X^{\alpha}}{\partial x^{\sigma}\partial x^{\tau}}}_{\Gamma^{\rho}_{\sigma\tau}} \qquad \text{eq (6)}$$

$$= \frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x^{\mu'}}\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\Gamma^{\rho}_{\sigma\tau}$$

Eq (6) is the transformation equation of a rank 3 tensor so we would say that the affine connection transforms like a tensor if we only had Term 1 in our transformation equation. However, our transformation equation has a second term:

$$\text{Term 2} = \left(\frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial X^{\alpha}}\right)\left(\frac{\partial X^{\alpha}}{\partial x^{\sigma}}\frac{\partial^2 X^{\sigma}}{\partial x^{\mu'}\partial x^{\nu'}}\right) \quad \text{but } \frac{\partial x^{\rho}}{\partial X^{\alpha}}\frac{\partial X^{\alpha}}{\partial x^{\sigma}} = \delta^{\rho}_{\sigma} \text{ so}$$

$$= \frac{\partial x^{\lambda'}}{\partial x^{\rho}}\delta^{\rho}_{\sigma}\frac{\partial^2 X^{\sigma}}{\partial x^{\mu'}\partial x^{\nu'}}$$

$$= \frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial^2 X^{\rho}}{\partial x^{\mu'}\partial x^{\nu'}} \qquad \text{eq (7)}$$

We can combine the results of eq (5), eq (6) and eq (7) to obtain:

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x^{\mu'}}\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\Gamma^{\rho}_{\sigma\tau} + \frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial^2 X^{\rho}}{\partial x^{\mu'}\partial x^{\nu'}} \qquad \text{eq (8)}$$

From this, we determine that **the affine connection is not a tensor**. Again, a second derivative term “ruins everything.” But this makes us wonder: Could we modify the way we take derivatives such that these second derivative terms cancel and result, ultimately, in a tensor. This, it turns out, is the motivation for the covariant derivative.

However, before we end this article, we can further simplify the term on the far righthand side of eq (8) by noting:

$$\frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial x^{\lambda'}} = \delta^{\lambda}_{\nu} \qquad \text{eq (9)}$$

Thus:

$$\frac{\partial x^{\lambda\prime}}{\partial x^{\rho}}\frac{\partial^2 X^{\rho}}{\partial x^{\mu'}\partial x^{\nu'}} = \frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial x^{\nu'}}\right) = 0$$

$$= \left(\frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x^{\mu'}\partial x^{\nu'}} + \frac{\partial^2 x^{\lambda'}}{\partial x^{\rho}\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial x^{\rho}}{\partial x^{\nu'}}\right) = 0 \quad \text{eq (10)}$$

Which means that:

$$\frac{\partial x^{\lambda'}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x^{\mu'} x^{\nu'}} = -\frac{\partial^2 x^{\lambda'}}{\partial x^{\rho} x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial x^{\rho}}{\partial x^{\nu'}} \qquad \text{eq (11)}$$

Substituting eq (11) into eq (8), we have:

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda\prime}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x^{\mu'}}\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\Gamma^{\rho}_{\sigma\tau} - \frac{\partial^2 x^{\lambda'}}{\partial x^{\rho} x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial x^{\rho}}{\partial x^{\nu'}}$$

This is just another way of writing eq (8).