This explanatory note is adapted from “Covariant Derivative” uploaded by Andrew Dotson, 11 June 2019, https://www.youtube.com/watch?v=TVeLh5LiEl8

Established in previous explanatory notes on this page were that:

1) Taking the derivative of a tensor does not necessarily yield another tensor. Instead, when we try to transform such a derivative, we come up with a part that looks like a correct tensor transformation plus additional second derivative terms like:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial X^{\alpha}} \frac{\partial^2 X^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \qquad \text{eq (1)}$$

We called such second derivative terms affine connections.

2). The affine connection is not a tensor. When we perform a coordinate transformation on the affine connection, we get:

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda'}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x^{\mu'}} \frac{\partial x^{\alpha}}{\partial x^{\nu'}} \Gamma^{\rho}_{\tau\alpha} - \frac{\partial x^{\alpha}}{\partial x^{\mu'}} \frac{\partial x^{\rho}}{\partial x^{\nu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^{\rho} x^{\alpha}} \qquad \text{eq (2)}$$

Our goal in this article is to modify the way we take derivatives such that the result is a tensor.

Let’s start by taking the derivative of a 4-vector, $\frac{\partial A^{\lambda'}}{\partial x^{\mu'}}$. We know how vectors transform:

$$A^{\lambda'} = \frac{\partial x^{\lambda'}}{\partial x^{\nu}} A^{\nu} \qquad \text{eq (3)}$$

Thus, we have:

$$\frac{\partial A^{\lambda'}}{\partial x^{\mu'}} = \frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^{\lambda'}}{\partial x^{\nu}}A^{\nu}\right)$$

$$= \frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial}{\partial x^{\sigma}}\left(\frac{\partial x^{\lambda'}}{\partial x^{\nu}}A^{\nu}\right)$$

$$= \underbrace{\frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^{\sigma}\partial x^{\nu}}A^{\nu}}_{\substack{\text{"Offending"}\\ \text{2nd}\\ \text{derivative}\\ \text{term}}} + \underbrace{\frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial x^{\lambda'}}{\partial x^{\nu}}\frac{\partial A^{\nu}}{\partial x^{\sigma}}}_{\substack{\text{Correct}\\ \text{transformation}\\ \text{for a rank 2}\\ \text{mixed tensor.}}} \qquad \text{eq (4)}$$

We'd like to define an operation, the covariant derivative, that will incorporate the affine connection and cancel the **offending 2nd derivative term** in eq (4)—an equation of the form:

$$D_{\mu}A^{\lambda'} \equiv \frac{\partial A^{\lambda'}}{\partial x^{\mu'}} + \Gamma^{\lambda'}_{\mu'\nu'}A^{\nu'} \qquad \text{eq (5)}$$

Eq (4) gives us an expression for $\frac{\partial A^{\lambda'}}{\partial x^{\mu'}}$. We substitute that into eq (5). Then we plug the value for $\Gamma^{\lambda'}_{\mu'\nu'}$ given in eq (2) into eq (5) and distribute it over *A* with unprimed indices:

$$\begin{aligned} D_{\mu}A^{\lambda'} = &\frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^{\sigma}\partial x^{\nu}}A^{\nu} + \frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial x^{\lambda'}}{\partial x^{\nu}}\frac{\partial A^{\nu}}{\partial x^{\sigma}} \\ &+\frac{\partial x^{\lambda\prime}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x^{\mu'}}\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\Gamma^{\rho}_{\tau\alpha}\frac{\partial x^{\nu'}}{\partial x^{\beta}}A^{\beta} - \frac{\partial x^{\alpha}}{\partial x^{\mu'}}\frac{\partial x^{\rho}}{\partial x^{\nu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^{\rho}x^{\alpha}}\frac{\partial x^{\nu'}}{\partial x^{\beta}}A^{\beta} \end{aligned} \qquad \text{eq (6)}$$

Next we need to simplify eq (6) by pulling out some delta functions. We can't find any in the first two terms, but we can in the last two terms.

$$D_\mu A^{\lambda'} = \frac{\partial x^\sigma}{\partial x^{\mu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^\sigma \partial x^\nu} A^\nu + \frac{\partial x^\sigma}{\partial x^{\mu'}} \frac{\partial x^{\lambda'}}{\partial x^\nu} \frac{\partial A^\nu}{\partial x^\sigma}$$
$$+ \frac{\partial x^{\lambda'}}{\partial x^\rho} \frac{\partial x^\tau}{\partial x^{\mu'}} \underbrace{\frac{\partial x^\alpha}{\partial x^{\nu'}} \Gamma^\rho_{\tau\alpha} \frac{\partial x^{\nu'}}{\partial x^\beta}}_{\delta^\alpha_\beta} A^\beta - \frac{\partial x^\alpha}{\partial x^{\mu'}} \underbrace{\frac{\partial x^\rho}{\partial x^{\nu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^\rho x^\alpha} \frac{\partial x^{\nu'}}{\partial x^\beta}}_{\delta^\rho_\beta} A^\beta \qquad \text{eq (7)}$$

The terms that make up the delta functions disappear from eq (7) and we contract the indices on the A^β terms to get:

$$= \frac{\partial x^\sigma}{\partial x^{\mu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^\sigma \partial x^\nu} A^\nu + \frac{\partial x^\sigma}{\partial x^{\mu'}} \frac{\partial x^{\lambda'}}{\partial x^\nu} \frac{\partial A^\nu}{\partial x^\sigma}$$
$$+ \frac{\partial x^{\lambda'}}{\partial x^\rho} \frac{\partial x^\tau}{\partial x^{\mu'}} \Gamma^\rho_{\tau\alpha} A^\alpha - \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^\rho x^\alpha} A^\rho \qquad \text{eq (8)}$$

rename indices : $\rho \to \nu$, $\tau \to \sigma$ rename indices : $\alpha \to \sigma$, $\rho \to \nu$

Next we rename dummy indices. When we do this, the first and fourth terms cancel:

$$= \frac{\partial x^\sigma}{\partial x^{\mu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^\sigma \partial x^\nu} A^\nu + \frac{\partial x^\sigma}{\partial x^{\mu'}} \frac{\partial x^{\lambda'}}{\partial x^\nu} \frac{\partial A^\nu}{\partial x^\sigma}$$
$$+ \frac{\partial x^\sigma}{\partial x^{\mu'}} \frac{\partial x^{\lambda'}}{\partial x^\nu} \Gamma^\rho_{\tau\alpha} A^\alpha - \frac{\partial x^\sigma}{\partial x^{\mu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^\sigma \partial x^\nu} A^\nu \qquad \text{eq (9)}$$

We factor out $\frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial x^{\lambda'}}{\partial x^{\nu}}$.That leaves us with:

$$D_{\mu'}A^{\nu'}=\frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial x^{\lambda'}}{\partial x^{\nu}}\left(\frac{\partial A^{\nu}}{\partial x^{\sigma}}+\Gamma^{\nu}_{\sigma\alpha}A^{\alpha}\right) \qquad \text{eq (10)}$$

Eq (11), with the index cancelations, shows us that the covariant derivative (the thing in parentheses in eq 10) does, indeed, transform as a tensor:

$$D_{\mu'}A^{\nu'}=\frac{\partial x^{\sigma}}{\partial x^{\mu'}}\frac{\partial x^{\lambda'}}{\partial x^{\nu}}\left(\frac{\partial A^{\nu}}{\partial x^{\sigma}}+\Gamma^{\nu}_{\sigma\alpha}A^{\alpha}\right) \qquad \text{eq (11)}$$