$$
\begin{aligned}
\nabla \times (\nabla \times \mathbf{V}) &= \nabla \times \left(\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \mathbf{k} \right) \\
&= \left(\frac{\partial}{\partial y} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \right) \mathbf{i} \\
&\quad + \left(\frac{\partial}{\partial z} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) - \frac{\partial}{\partial x} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \right) \mathbf{j} \\
&\quad + \left(\frac{\partial}{\partial x} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \right) \mathbf{k} \\
&= \left(\frac{\partial^2 V_y}{\partial y \partial x} - \frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_z}{\partial z \partial x} \right) \mathbf{i} \\
&\quad + \left(\frac{\partial^2 V_z}{\partial z \partial y} - \frac{\partial^2 V_y}{\partial z^2} - \frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_x}{\partial x \partial y} \right) \mathbf{j} \\
&\quad + \left(\frac{\partial^2 V_x}{\partial x \partial z} - \frac{\partial^2 V_z}{\partial x^2} - \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_y}{\partial y \partial z} \right) \mathbf{k} \\
&= \left(\left(\frac{\partial^2 V_y}{\partial y \partial x} + \frac{\partial^2 V_z}{\partial z \partial x} \right) - \left(\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) \right) \mathbf{i} \\
&\quad + \left(\left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial z \partial y} \right) - \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial z^2} \right) \right) \mathbf{j} \\
&\quad + \left(\left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} \right) - \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} \right) \right) \mathbf{k} \\
&= \left(\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y \partial x} + \frac{\partial^2 V_z}{\partial z \partial x} \right) - \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) \right) \mathbf{i} \\
&\quad + \left(\left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial z \partial y} \right) - \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2} \right) \right) \mathbf{j} \\
&\quad + \left(\left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial z^2} \right) - \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \right) \right) \mathbf{k} \\
&= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) \\
&\quad - \left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(V_x \mathbf{i} + V_y \mathbf{j} + V_z \mathbf{k} \right) \right) \\
&= \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}
\end{aligned}
$$